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Group Completing Monoidal Categories

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A monoidal category is a category $\mathcal{M}$ together with a coherently associative bifunctor $\square: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ which has a two-sided identity object $*$. $\mathcal{M}$ is a strict monoidal category if $\square$ is strictly associative; any monoidal category is naturally equivalent to a strict monoidal category [4]. Henceforth we assume all monoidal categories to be strict. A monoidal category $\mathcal{M}$ is symmetric if there is a natural isomorphism of functors $c: \square \to \square T$ that makes $\square$ coherently commutative (where T is the functor that interchanges factors in $\mathcal{M} \times \mathcal{M}$). A symmetric strict monoidal category is said to be permutative. Our prime example is the category of finitely generated projective left modules over a ring R under direct sum, which we denote $\mathcal{P}_R$.

The Grothendieck group of a small permutative category $\mathcal{M}$, $K_0\mathcal{M}$, is the group completion of the monoid of isomorphism classes of objects of $\mathcal{M}$ under $\square$; $K_0\mathcal{P}_R$ is by definition K_0R in the algebraic K-theory of the ring R. Let $\Omega\mathcal{M}$ be the set of pairs (A, f) where A is an object of $\mathcal{M}$ and f is an automorphism of A. The Whitehead group of $\mathcal{M}$ is a group $K_1\mathcal{M}$ supplied with a map from the set $\Omega\mathcal{M}$ to $K_1\mathcal{M}$ that is universal for maps γ from $\Omega\mathcal{M}$ to an Abelian group that satisfy:

(i) If $j: A \to B$ is an isomorphism and $jf = gj$, then $\gamma(A, f) = \gamma(B, g)$;

(ii) $\gamma(A \square B, f \square g) = \gamma(A, f) + \gamma(B, g)$;

(iii) $\gamma(A, fh) = \gamma(A, f) + \gamma(A, h)$;

where (A, f), (B, g), and (A, h) are in $\Omega\mathcal{M}$. $K_1\mathcal{P}_R$ is K_1R in the algebraic K-theory of R. For a more complete discussion of these definitions, see [2] and [5].

Given a small monoidal category $\mathcal{M}$ we will construct a monoidal category $K\mathcal{M}$ whose objects form a group under $\square$ that is isomorphic to the group completion of the monoid of objects of $\mathcal{M}$ under $\square$. We call $K\mathcal{M}$ the group completion of $\mathcal{M}$. If $\mathcal{M}$ is permutative, the set of isomorphism classes of objects of $K\mathcal{M}$ will form a group isomorphic to $K_0\mathcal{M}$; if $\mathcal{M}$ is also skeletal, the automorphism group of any object of $K\mathcal{M}$ will be shown to be isomorphic to $K_1\mathcal{M}$. These facts show the inevitability of the appearance of the functor K_1, and they clarify the roles of composition and $\square$ in the definition of K_1. They also indicate the difficulty of using algebraic or categorical techniques to investigate higher K-groups.

In any small monoidal category $\mathcal{G}$, the set of morphisms of $\mathcal{G}$, Mor $\mathcal{G}$, forms a monoid under $\square$. If the objects of $\mathcal{G}$, Ob $\mathcal{G}$, form a group under $\square$, Ob $\mathcal{G}$ acts on Mor $\mathcal{G}$ by conjugation. Let A denote both an object and its identity map, and let $\bar{A}$ denote the inverse object of A; then, if $f \in \operatorname{Mor} \mathcal{G}$, $A(f) = A \square f \square \bar{A}$. Star $*$ is the set of all morphisms of $\mathcal{G}$ with domain $*$. It is a submonoid of Mor $\mathcal{G}$ and is closed under the action of Ob $\mathcal{G}$.

PROPOSITION 1. *Let $\mathcal{G}$ be a small monoidal category with* Ob $\mathcal{G}$ *a group under* $\square$. *Then*

(i) $\square$ *and composition agree in* $\hom_{\mathcal{G}}(*, *)$ *and are commutative;*

(ii) $\hom_{\mathcal{G}}(A, A)$ *and* $\hom_{\mathcal{G}}(*, *)$ *are isomorphic as monoids under composition;*

(iii) Mor $\mathcal{G}$ *is isomorphic to the semidirect product of* Star $*$ *and* Ob $\mathcal{G}$ *as a monoid under* $\square$; *and*

(iv) *a morphism of $\mathcal{G}$ is $\square$-invertible if and only if it is an isomorphism.*

Proof. Part (i) is immediate since $\square$ and composition have common identities and distribute over each other in $\hom_{\mathcal{G}}(*, *)$ [6, p. 43]. The assignment to f in $\hom_{\mathcal{G}}(A, A)$ of $f \square \bar{A}$ in $\hom_{\mathcal{G}}(*, *)$ defines the isomorphism of part (ii). The semidirect product of Star $*$ and Ob $\mathcal{G}$ is the cartesian product with multiplication given by $(f, A)(g, B) = (f \square A(g), A \square B) = (f \square A \square g \square \bar{A}, A \square B)$. The assignment to $f\colon A \to B$ of $(f \square \bar{A}, A)$ is easily seen to be an isomorphism between Mor $\mathcal{G}$ and the semidirect product. If $f\colon A \to B$ is an isomorphism, then $\bar{A} \square f^{-1} \square \bar{B}$ is the $\square$-inverse for f:

$$\begin{aligned}(\bar{A} \square f^{-1} \square \bar{B}) \square f &= ((\bar{A} \square f^{-1} \square \bar{B}) \square B)(\bar{A} \square f)\\ &= (\bar{A} \square f^{-1})(\bar{A} \square f)\\ &= \bar{A} \square A\\ &= *.\end{aligned}$$

Conversely if $g\colon \bar{A} \to \bar{B}$ is a $\square$-inverse for f, then $f^{-1} = A \square g \square B$ since

$$\begin{aligned}(A \square g \square B) f &= (A \square g \square B)(A \square \bar{A} \square f)\\ &= A \square (g \square B)(\bar{A} \square f)\\ &= A \square g \square f\\ &= A \square *\\ &= A.\end{aligned}$$

Note that two naturally isomorphic functors from a skeletal groupoid to $\mathscr{G}$ which agree on objects must be equal by parts (i) and (ii) of the proposition. In particular if $\mathscr{G}$ is a skeletal permutative groupoid, $\square$ is strictly commutative.

Given a small monoidal category $\mathscr{M}$ and a surjective map of monoids $\sigma\colon \operatorname{Ob} \mathscr{M} \to N$ we now construct a monoidal category $V_\sigma \mathscr{M}$ and a monoidal functor $\mathscr{M} \to V_\sigma \mathscr{M}$ which is universal for monoidal functors whose object maps have σ as a right factor. Form the category $U_\sigma \mathscr{M}$ [3] by letting the objects of $U_\sigma \mathscr{M}$ be the elements of N and the maps from α to β in $U_\sigma \mathscr{M}$ be sequences (nonempty if $\alpha \neq \beta$) $x_n \cdots x_1$ of morphisms of $\mathscr{M}$ such that:

(i) $\sigma(\text{domain } x_1) = \alpha, \sigma(\text{domain } x_{i+1}) = \sigma(\text{range } x_i)$, and $\sigma(\text{range } x_n) = \beta$;

(ii) the x_i are nonidentity morphisms of $\mathscr{M}$; and

(iii) the composition of x_{i+1} and x_i is not defined in $\mathscr{M}$.

If $x_n \cdots x_1\colon \alpha \to \beta$ and $y_m \cdots y_1\colon \beta \to \gamma$ in $U_\sigma \mathscr{M}$, their composition is the unique sequence satisfying (i), (ii), and (iii) obtained from $y_m \cdots y_1 x_n \cdots x_1$ by performing all possible compositions in $\mathscr{M}$ and deleting identities. The empty sequence from α to α serves as the identity in $U_\sigma \mathscr{M}$. There is an obvious functor $\mathscr{M} \to U_\sigma \mathscr{M}$ which is universal for functors whose object maps have σ as a right factor. If p and q are morphisms of $U_\sigma \mathscr{M}$, define $p \sim q$ if $p = w(x \square y) z$ and $q = w(x \square D)(A \square y) z$ or $q = w(B \square y)(x \square C) z$ where w, z are morphisms of $U_\sigma \mathscr{M}$; x, y are morphisms of $\mathscr{M}$; and A, B, C, D are objects of $\mathscr{M}$ such that $\sigma(\text{domain } x) = \sigma(A)$, $\sigma(\text{range } x) = \sigma(B)$, $\sigma(\text{domain } y) = \sigma(C)$, and $\sigma(\text{range } y) = \sigma(D)$. $\sim$ generates an equivalence relation in $U_\sigma \mathscr{M}$ that preserves composition so the equivalence classes of morphisms form a category $V_\sigma \mathscr{M}$ with the same objects as $U_\sigma \mathscr{M}$. A morphism f of $V_\sigma \mathscr{M}$ is invertible if and only if every morphism of $\mathscr{M}$ in any sequence of $U_\sigma \mathscr{M}$ which represents f is invertible.

THEOREM 2. *$V_\sigma \mathscr{M}$ is a monoidal category.*

Proof. Given an object A of $\mathscr{M}$, we have a functor $\square A\colon U_\sigma \mathscr{M} \to U_\sigma \mathscr{M}$ which on objects is right multiplication by $\sigma(A)$ and on morphisms sends $x_n \cdots x_1$ to $(x_n \square A) \cdots (x_1 \square A)$. Since $\square A$ preserves the equivalence relation $\sim$, it induces a functor $\square A\colon V_\sigma \mathscr{M} \to V_\sigma \mathscr{M}$. If B is another object of $\mathscr{M}$ with $\sigma(B) = \sigma(A)$, $\square B$ equals $\square A$ on $V_\sigma \mathscr{M}$ because if $f\colon C \to D$ in $\mathscr{M}$, $f \square A \sim (f \square B)(C \square B) = f \square B$ in $U_\sigma \mathscr{M}$. Thus if α is an object of $V_\sigma \mathscr{M}$ we obtain a functor $\square \alpha\colon V_\sigma \mathscr{M} \to V_\sigma \mathscr{M}$ induced by $\square A$ where A is any object in $\mathscr{M}$ with $\sigma(A) = \alpha$. We can define $\alpha \square\colon V_\sigma \mathscr{M} \to V_\sigma \mathscr{M}$ similarly. We now use these functors to define an associative bifunctor $\square$: $V_\sigma \mathscr{M} \times V_\sigma \mathscr{M} \to V_\sigma \mathscr{M}$. If $x_1\colon \alpha_1 \to \alpha_2$ and $y_1\colon \beta_1 \to \beta_2$ in $V_\sigma \mathscr{M}$, define $x_1 \square y_1\colon \alpha_1 \square \beta_1 \to \alpha_2 \square \beta_2$ to be $(x_1 \square \beta_2)(\alpha_1 \square y_1)$. To show $\square$ is a functor, suppose $x_2\colon \alpha_2 \to \alpha_3$ and $y_2\colon \beta_2 \to \beta_3$ in $V_\sigma\mathcal{M}$; then

$$(x_2 \Box y_2)(x_1 \Box y_1) = (x_2 \Box \beta_3)(\alpha_2 \Box y_2)(x_1 \Box \beta_2)(\alpha_1 \Box y_1)$$

and

$$\begin{aligned} x_2x_1 \Box y_2y_1 &= (x_2x_1 \Box \beta_3)(\alpha_1 \Box y_2y_1) \\ &= (x_2 \Box \beta_3)(x_1 \Box \beta_3)(\alpha_1 \Box y_2)(\alpha_1 \Box y_1). \end{aligned}$$

Therefore, we must show $(\alpha_2 \Box y_2)(x_1 \Box \beta_2) = (x_1 \Box \beta_3)(\alpha_1 \Box y_2)$, but this relation is clearly true for x_1 and y_2, actually morphisms of $\mathcal{M}$, so it holds in $V_\sigma\mathcal{M}$. Associativity follows from the associativity of $\Box$ on $\mathcal{M}$, and the identity element $*$ of N is a two-sided identity for $\Box$.

The natural functor $\hat{\sigma}\colon \mathcal{M} \to V_\sigma\mathcal{M}$ is necessarily monoidal. To show $\hat{\sigma}$ has the required universal property, let $F\colon \mathcal{M} \to \mathcal{N}$ be a monoidal functor whose object map has σ as a right factor; then there is a unique functor $F_1\colon U_\sigma\mathcal{M} \to \mathcal{N}$ which factors F and thus preserves the equivalence relation $\sim$. Hence F_1 induces a unique monoidal functor $F_2\colon V_\sigma\mathcal{M} \to \mathcal{N}$ such that $F_2\hat{\sigma} = F$.

If $\mathcal{M}$ is permutative, we extend the equivalence relation on morphisms of $U_\sigma\mathcal{M}$ by $p \sim q$ if $p = x(R \Box c(A, B) \Box S)y$ and $q = x(R \Box c(A', B') \Box S)y$ where x, y are morphisms in $U_\sigma\mathcal{M}$ and A, B, A', B', R, and S are objects of $\mathcal{M}$ such that $\sigma(A) = \sigma(A')$ and $\sigma(B) = \sigma(B')$. Then $V_\sigma\mathcal{M}$ will be permutative and $\hat{\sigma}\colon \mathcal{M} \to V_\sigma\mathcal{M}$ will be universal for permutative functors whose object maps have σ as a right factor.

COROLLARY 3. *The category of small monoidal (or permutative) categories is complete and cocomplete.*

Proof. Products and coproducts exist by [1]. Equalizers are trivial so we need only show the existence of coequalizers. Let $F, G\colon \mathcal{M} \to \mathcal{N}$ and let P be the coequalizer of the vertex maps in the category of small monoids. Since the natural map $\sigma\colon \mathrm{Ob}\,\mathcal{N} \to P$ is surjective, $V_\sigma\mathcal{N}$ is a monoidal category. If p and q are morphisms of $V_\sigma\mathcal{N}$, define $p \sim q$ if $p = x(\alpha \Box F(u) \Box \beta)y$ and $q = x(\alpha \Box G(u) \Box \beta)y$ where x and y are morphisms of $V_\sigma\mathcal{N}$, α and β are objects of $V_\sigma\mathcal{N}$, and u is a morphism of $\mathcal{M}$. $\sim$ generates an equivalence relation preserving composition and $\Box$. The category of equivalence classes $\mathcal{G}$ is monoidal and thus the desired coequalizer.

The (non-Abelian) group completion of a monoid M is usually constructed by taking the free group on M modulo the normal subgroup generated by elements of the form $(a \cdot b)\,b^{-1}a^{-1}$, a and b in M (where we denote multiplication in M by $\cdot$ for clarity). We give another construction. Let $\bar{M}$ be M with the opposite multiplication; if a is in M, $\bar{a}$ denotes the corresponding element of $\bar{M}$. Define the group completion $K(M)$ to be the free product of monoids

$M \times \bar{M}$ modulo the equivalence relation generated by demanding $a\bar{a} \sim 1 \sim \bar{a}a$. We can imitate this construction for a monoidal category $\mathscr{M}$. Let $\bar{\mathscr{M}}$ be $\mathscr{M}$ with the opposite product. The objects of the coproduct $\mathscr{M} * \bar{\mathscr{M}}$ are exactly the elements of the free product of monoids $\operatorname{Ob}\mathscr{M} * \operatorname{Ob}\bar{\mathscr{M}}$ so there is a surjection of monoids $\sigma\colon \operatorname{Ob}(\mathscr{M} * \bar{\mathscr{M}}) \to K(\operatorname{Ob}\mathscr{M})$. Define the group completion of $\mathscr{M}$, $K\mathscr{M}$, to be $V_\sigma(\mathscr{M} * \bar{\mathscr{M}})$. The objects of $K\mathscr{M}$ are exactly the elements of the group $K(\operatorname{Ob}\mathscr{M})$, and there is a natural monoidal functor $H\colon \mathscr{M} \to K\mathscr{M}$. H has the universal property that if $F\colon \mathscr{M} \to \mathscr{G}$ is a monoidal functor into a monoidal category $\mathscr{G}$ whose objects form a group then there is a unique monoidal functor $F\colon K\mathscr{M} \to \mathscr{G}$ such that $F = FH$. The construction remains the same if $\mathscr{M}$ is permutative.

PROPOSITION 4. *Let F and G be monoidal functors from a monoidal category $\mathscr{M}$ to a monoidal groupoid $\mathscr{N}$.* If $\eta\colon F \to G$ *is an additive natural transformation of functors* ($\eta(A \,\square\, B) = \eta A \,\square\, \eta B$) *then there exists an additive natural transformation* $K\eta\colon KF \to KG$.

Proof. Additive natural transformations between functors from $\mathscr{M}$ to $\mathscr{N}$ are in one-to-one correspondence with monoidal functors from $\mathscr{M}$ to $\mathscr{F}(\Delta[1], \mathscr{N})$ where $\mathscr{F}$ is the functor category and $\Delta[1]$ is the category with two objects 0 and 1 and one nonidentity morphism from 0 to 1. Thus η induces a monoidal functor from $\mathscr{M}$ to $\mathscr{F}(\Delta[1], K\mathscr{N})$. Since $\mathscr{N}$ is a groupoid, $K\mathscr{N}$ is a groupoid; therefore, every morphism of $K\mathscr{N}$ has a $\square$-inverse by Proposition 1. Then the objects of $\mathscr{F}(\Delta[1], K\mathscr{N})$ form a group under $\square$ so we obtain a monoidal functor from $K\mathscr{M}$ to $\mathscr{F}(\Delta[1], K\mathscr{N})$ which yields the desired additive natural transformation.

If $\mathscr{M}$ is a monoidal category, let $\operatorname{Iso}\mathscr{M}$ denote the subcategory of $\mathscr{M}$ consisting of the same objects and the isomorphisms of $\mathscr{M}$.

THEOREM 5. *There is a natural isomorphism* $K(\operatorname{Iso}\mathscr{M}) \to \operatorname{Iso} K\mathscr{M}$.

Proof. $K(\operatorname{Iso}\mathscr{M})$ is a groupoid so $K(\operatorname{Iso}\mathscr{M}) \to \operatorname{Iso} K\mathscr{M}$. A morphism f in $K\mathscr{M}$ is invertible if and only if all representatives of f in $U_\sigma(\mathscr{M} * \bar{\mathscr{M}})$ are invertible; a morphism $x_n \cdots x_1$ of $U_\sigma(\mathscr{M} * \bar{\mathscr{M}})$ is invertible if and only if each x_i is in $\operatorname{Iso}(\mathscr{M} * \bar{\mathscr{M}}) \approx (\operatorname{Iso}\mathscr{M}) * (\operatorname{Iso}\bar{\mathscr{M}})$. Hence, f is in the image of $K(\operatorname{Iso}\mathscr{M}) \to \operatorname{Iso} K\mathscr{M}$.

THEOREM 6. *If $\mathscr{M}$ is a skeletal permutative category, then the group of isomorphism classes of objects of $K\mathscr{M}$ is isomorphic to $K_0\mathscr{M}$ and the group of automorphisms of any object of $K\mathscr{M}$ is isomorphic to $K_1\mathscr{M}$.*

Proof. The first part is clear from Theorem 5 and the definitions. To prove the second part, we need only show $\operatorname{Aut}_{K\mathscr{M}}(*, *)$ isomorphic to $K_1\mathscr{M}$

by Proposition 1. Let $K_1\mathcal{M}$ be the monoidal category with one object and one morphism for each element of $K_1\mathcal{M}$; composition and $\square$ in $K_1\mathcal{M}$ are given by the group operation in $K_1\mathcal{M}$. There is a monoidal functor from Iso $\mathcal{M}$ to $K_1\mathcal{M}$ which sends $f: A \to A$ to the equivalence class of (A, f). Since the objects of $K_1\mathcal{M}$ form the trivial group, we obtain a monoidal functor from $K(\text{Iso}\,\mathcal{M}) = \text{Iso}\,K\mathcal{M}$ to $K_1\mathcal{M}$ which gives us a function from $\text{Aut}_{K\mathcal{M}}(*, *)$ to $K_1\mathcal{M}$. On the other hand, $\Omega\mathcal{M}$ maps to $\text{Aut}_{K\mathcal{M}}(*, *)$ by sending the pair (A, f) to $f \square \bar{A}$. If (B, g) is in $\Omega\mathcal{M}$ and $j: A \to B$ is an isomorphism in $\mathcal{M}$ such that $jf = gj$, then

$$f \square \bar{A} = (j \square \bar{j})(f \square \bar{A}) = jf \square \bar{j} = gj \square \bar{j} = (g \square \bar{B})(j \square \bar{j}) = g \square \bar{B}.$$

Also

$$f \square \bar{A} \square g \square \bar{B} = f \square c(*, \bar{A})((g \square \bar{B}) \square \bar{A}) = f \square g \square \bar{B} \square \bar{A}$$

since $\mathcal{M}$ is permutative. If (A, f) and (A, h) are in $\Omega\mathcal{M}$,

$$fh \square \bar{A} = (f \square \bar{A})(h \square \bar{A}) = f \square \bar{A} \square h \square \bar{A}$$

by Proposition 1. Hence, by the universal property of $K_1\mathcal{M}$ we obtain a homomorphism $K_1\mathcal{M} \to \text{Aut}_{K\mathcal{M}}(*, *)$. The two maps thus obtained must necessarily be inverses.

COROLLARY 7. *If $\mathcal{M}$ is a skeletal permutative groupoid and $\mathcal{G}$ is any skeleton of $K\mathcal{M}$, then* Ob $\mathcal{G} = K_0\mathcal{M}$ *and* Mor $\mathcal{G} = K_0\mathcal{M} \oplus K_1\mathcal{M}$.

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